Ward Identities in the Derivation of Hawking Radiation from Anomalies

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Robinson and Wilczek suggested a new method of deriving Hawking radiation by the consideration of anomalies. The basic idea of their approach is that the flux of Hawking radiation is determined by anomaly cancellation conditions in the Schwarzschild black hole (BH) background. Iso et al. extended the method to a charged Reissner-Nordström BH and a rotating Kerr BH, and they showed that the flux of Hawking radiation can also be determined by anomaly cancellation conditions and regularity conditions of currents at the horizon. Their formulation gives the correct Hawking flux for all the cases at infinity and thus provides a new attractive method of understanding Hawking radiation. We present some arguments clarifying for this derivation. We show that the Ward identities and boundary conditions for covariant currents without referring to the Wess-Zumino terms and the effective action are sufficient to derive Hawking radiation. Our method, which does not use step functions, thus simplifies some of the technical aspects of the original formulation.

§1. Introduction

Hawking radiation is derived by using the quantum effect in black hole (BH) physics. There are several methods of deriving of Hawking radiation.[1)–3)] Hawking's original derivation, which calculates the Bogoliubov coefficients between the in- and out-states for a body collapsing to form a BH, is very direct and physical.[1)] It is well-known that the Hawking flux agrees with the blackbody flux at the temperature $T = \kappa/2\pi$, where κ is the surface gravity of a BH, if we ignore the backscattering of particles falling into the horizon, i.e., the gray body radiation.

Robinson and Wilczek demonstrated a new method of deriving of Hawking radiation.[4)] They derived Hawking radiation by the consideration of quantum anomalies. Their derivation has an important advantage in localizing the source of Hawking radiation near the horizon where anomalies are visible. Since both of two anomalies and Hawking radiation are typical quantum effects, it is natural that Hawking radiation is related to the anomalies in their derivation. Iso et al. improved the approach of Ref. 4) and extended the method to a charged Reissner-Nordström BH.[5)] This approach was also extended to a rotating Kerr BH and a charged and rotating Kerr-Newman BH by Murata and Soda[6)] and by Iso et al.[7)]

The approach of Iso et al.[7)] is very transparent and interesting. However, there remain several points to be clarified. First, Iso et al. start by using both the consistent and covariant currents. However, they only impose boundary conditions on covariant currents. As discussed in Ref. 5), it is not clear why we should use

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covariant currents instead of consistent ones to specify the boundary conditions at the horizon. Banerjee and Kulkarni considered an approach using only covariant currents without consistent currents.[8] However, their approach heavily relies on the Wess-Zumino terms defined by consistent currents.[9] The Wess-Zumino terms are also used in the approach of Iso et al. Therefore, Banerjee and Kulkarni's approach is not completely described by covariant currents only.

Second, in Iso et al.'s approach the region outside the horizon must be divided into two regions because the effective theories are different near and far from the horizon. They thus used step functions to divide these two regions. We think that the region near the horizon and the region far from the horizon are continuously related. Nevertheless, if one uses step functions, terms with delta functions appear that originate from the derivatives of step functions when one considers the variation of the effective action. They disregarded the terms without delta functions by claiming that these terms correspond to the contributions of the ingoing mode. This is the second issue that we wish to address here. Banerjee and Kulkarni also considered an approach without step functions.[10] They obtained the Hawking flux by using the effective actions and two boundary conditions for covariant currents. However, they assumed that the effective actions are 2-dimensional in both the region near the horizon and that far from the horizon.[11),12] As discussed in Iso et al.'s approach, the original 4-dimensional theory is the 2-dimensional effective theory in the region near the horizon. However, the effective theory should be 4-dimensional in the region far from the horizon.

In contrast with the above approaches, we derive the Hawking flux using only the Ward identities and two boundary conditions for the covariant currents. We formally perform the path integral, and the Nöther currents are constructed by the variational principle. Therefore, we can naturally treat the covariant currents.[13] We do not use the Wess-Zumino term, the effective action or step functions. Therefore, we do not need to define consistent currents. Although we use the two boundary conditions used in Banerjee and Kulkarni's method, we use the 4-dimensional effective theory far from the horizon and the 2-dimensional theory near the horizon. In this sense, our method corresponds to Iso et al.'s method. It is easier to understand the derivation of the Ward identities directly from the variation of matter fields than their derivation from the effective action since we consider Hawking radiation as resulting from the effects of matter fields.

Our approach is essentially based on Iso et al.'s approach. However, we simplify the derivation of Hawking radiation by clarifying the above issues. We only use the Ward identities and two boundary conditions for covariant currents, and we do not use the Wess-Zumino terms, the effective action or step functions.

The content of the paper is as follows. In §2, we show our simple derivation of Hawking radiation in a rotating Kerr BH background. Section 3 is devoted to conclusions and discussions. In Appendix A we show how to derive the Hawking flux in a charged Reissner-Nordstöm BH using our approach.

§2. Simple derivation

In this section, by clarifying the arguments used in the derivation in Ref. 7), we show that we can obtain the same result by a simplified method. Since we consider a rotating Kerr BH, the external space is given by the Kerr metric

$$\begin{aligned} ds^2 =& \frac{\triangle - a^2\sin^2\theta}{\Sigma^2}dt^2 + \frac{2a\sin^2\theta}{\Sigma^2}(r^2+a^2-\triangle)dtd\varphi \\ &+ \frac{a^2\triangle\sin^2\theta-(r^2+a^2)^2}{\Sigma^2}\sin^2\theta d\varphi^2 - \frac{\Sigma^2}{\triangle}dr^2 - \Sigma^2 d\theta^2, \end{aligned} \tag{2·1}$$

where $a \equiv J/M$, $\triangle \equiv r^2 - 2Mr + a^2 = (r-r_+)(r-r_-)$ and $\Sigma^2 \equiv r^2 + a^2\cos^2\theta$. $r_{+(-)}$ is the outer (inner) horizon. We consider quantum fields in the vicinity of the Kerr BH. In 4 dimensions, the action for a scalar field is given by

$$S = \frac{1}{2}\int d^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + S_{\rm int}, \tag{2·2}$$

where the first term is the kinetic term and the second term $S_{\rm int}$ represents the mass, potential and interaction terms. Note that the $U(1)$ gauge field does not appear in the Kerr BH background. This is a crucial difference from the charged BH. By performing the partial wave decomposition of ϕ in terms of the spherical harmonics ($\phi = \sum_{l,m}\phi_{lm}Y_{lm}$) and using the properties of metrics at the horizon, the action S near the horizon is written as[7)]

$$S = -\frac{1}{2}\sum_{l,m}\int dtdr(r^2+a^2)\phi^*_{lm}\left[\frac{(r^2+a^2)}{\triangle}\left(\partial_t + \frac{ima}{r^2+a^2}\right)^2 - \partial_r\frac{\triangle}{r^2+a^2}\partial_r\right]\phi_{lm}, \tag{2·3}$$

where we ignore $S_{\rm int}$ because the kinetic term dominates near the horizon in high-energy theory. From this action we find that ϕ_{lm} can be considered as $(1+1)$-dimensional complex scalar fields in the backgrounds of the dilaton Φ, metric $g_{\mu\nu}$ and $U(1)$ gauge field A_μ, which are defined by

$$\Phi = r^2 + a^2, \tag{2·4}$$

$$g_{tt} = f(r),\ g_{rr} = -\frac{1}{f(r)},\ g_{rt} = 0, \tag{2·5}$$

$$A_t = -\frac{a}{r^2+a^2},\ A_r = 0, \tag{2·6}$$

where $f(r) \equiv \triangle/(r^2+a^2)$. The $U(1)$ charge of the 2-dimensional field ϕ_{lm} is m.

From (2·3), we find that the effective theory is the $(1+1)$-dimensional theory near the horizon. However, we cannot simply regard the effective theory far from the horizon as the $(1+1)$-dimensional theory. We need to divide the region outside the horizon into two regions because the effective theories are different near the horizon and far from the horizon. We define region O as the region far from the horizon and region H as the region near the horizon. Note that the action in region O is $S_{(O)}[\phi, g^{\mu\nu}_{(4)}]$ and the action in region H is $S_{(H)}[\phi, g^{\mu\nu}_{(2)}, A_\mu, \Phi]$.

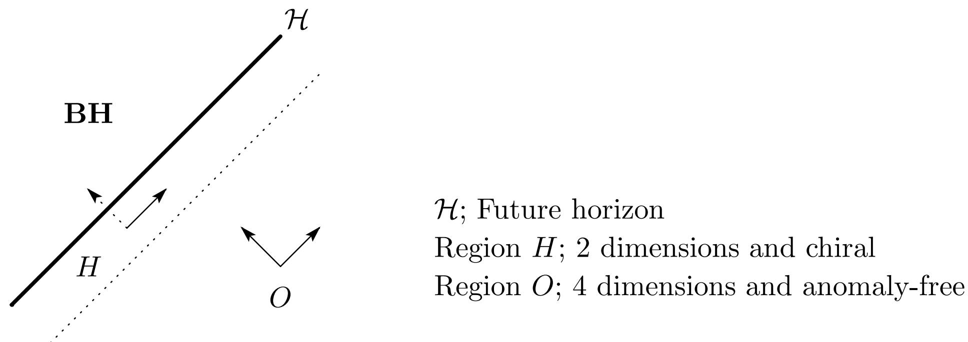

Fig. 1 Part of the Penrose diagram relevant to our analysis.

The dashed arrow in region H represents the ignored ingoing mode falling toward the horizon.

We can divide particles into ingoing modes falling toward the horizon (left-handed) and outgoing modes moving away from the horizon (right-handed) using a Penrose diagram[4),5),7)] (Fig. 1). Since the horizon is a null hypersurface, none of the ingoing modes at the horizon are expected to affect the classical physics outside the horizon. Thus, we ignore the ingoing modes. Therefore, anomalies appear with respect to the gauge or general coordinate symmetries since the effective theory is chiral near the horizon. Here, we do not consider the backscattering of ingoing modes, i.e., the gray body radiation.

We now present the derivation of Hawking radiation for the Kerr BH. First, we consider the effective theory in region O. The effective theory is 4-dimensional in region O, which we cannot reduce to a 2-dimensional theory. In contrast with the case of a charged BH, a 4-dimensional gauge field such as the Coulomb potential $A = -Q/r$ does not exist in a rotating Kerr BH. Therefore, we do not define the $U(1)$ gauge current in region O. In contrast, the effective theory in region H is a 2-dimensional chiral theory and we can regard part of the metric as a gauge field such as (2·6), since the action of (2·3) is $S_{(H)}[\phi, g^{\mu\nu}_{(2)}, A_\mu, \Phi]$.

Second, we consider the Ward identity for the gauge transformation in region H near the horizon. Here, we pretend to formally perform the path integral for $S_{(H)}[\phi, g^{\mu\nu}_{(2)}, A_\mu, \Phi]$, where the Nöther current is constructed by the variational principle, although we do not perform an actual path integral. Therefore, we can naturally treat *covariant* currents.[13)] As a result, we obtain the Ward identity with a gauge anomaly

$$\nabla_\mu J^\mu_{(H)} - \mathcal{C} = 0, \qquad (2\cdot 7)$$

where we define covariant currents $J^\mu_{(H)}(r)$ and $\mathcal{C}$ is a covariant gauge anomaly. This Ward identity is for right-handed fields. The covariant form of the 2-dimensional abelian anomaly $\mathcal{C}$ is given by

$$\mathcal{C} = \pm \frac{m^2}{4\pi\sqrt{-g_{(2)}}} \epsilon^{\mu\nu} F_{\mu\nu}, \qquad (\mu, \nu = t, r) \qquad (2\cdot 8)$$

where $+(-)$ corresponds to right(left)-handed matter fields, $\epsilon^{\mu\nu}$ is an antisymmetric tensor with $\epsilon^{tr} = 1$ and $F_{\mu\nu}$ is the field-strength tensor. Using the 2-dimensional metric (2·5), (2·7) is written as

$$\partial_r J^r_{(H)}(r) = \frac{m^2}{2\pi}\partial_r A_t(r). \tag{2·9}$$

By integrating Eq. (2·9) over r from r_+ to r, we obtain

$$J^r_{(H)}(r) = \frac{m^2}{2\pi}\left[A_t(r) - A_t(r_+)\right], \tag{2·10}$$

where we use the condition

$$J^r_{(H)}(r_+) = 0. \tag{2·11}$$

Condition (2·11) corresponds to the statement that free falling observers see a finite amount of the charged current at the horizon, i.e., (2·11) is derived from the regularity of covariant currents (see the appendix of Ref. 7)). We regard (2·10) as a covariant $U(1)$ gauge current appearing in region H near the horizon.

Third, we consider the Ward identity for the general coordinate transformation in region O far from the horizon. By improving the approach of Ref. 5), we define the formal 2-dimensional energy-momentum tensor in region O from the exact 4-dimensional energy-momentum tensor in region O to connect the thus-defined 2-dimensional energy-momentum tensor in region O with the 2-dimensional energy-momentum tensor in region H. Since the action is $S_{(O)}[\phi, g^{\mu\nu}_{(4)}]$ in region O, the Ward identity for the general coordinate transformation is written as

$$\nabla_\nu T^{\mu\nu}_{(4)} = 0, \tag{2·12}$$

where $T^{\mu\nu}_{(4)}$ is the 4-dimensional energy-momentum tensor. Since the Kerr background is stationary and axisymmetric, the expectation value of the energy-momentum tensor in the background depends only on r and θ, i.e., $\langle T^{\mu\nu}\rangle = \langle T^{\mu\nu}(r,\theta)\rangle$. The $\mu = t$ component of the conservation law (2·12) is written as

$$\partial_r(\sqrt{-g}T_{t(4)}^{\ r}) + \partial_\theta(\sqrt{-g}T_{t(4)}^{\ \theta}) = 0, \tag{2·13}$$

where $\sqrt{-g} = (r^2 + a^2\cos^2\theta)\sin\theta$. By integrating Eq. (2·13) over the angular coordinates θ and φ, we obtain

$$\partial_r T_{t(2)}^{\ r} = 0, \tag{2·14}$$

where we define the effective 2-dimensional tensor $T_{t(2)}^{\ r}$ by

$$T_{t(2)}^{\ r} \equiv \int d\Omega_{(2)}(r^2 + a^2\cos^2\theta)T_{t(4)}^{\ r}. \tag{2·15}$$

We define $T_{t(2)}^{\ r} \equiv T_{t(O)}^{\ r}$ to emphasize region O far from the horizon. The energy-momentum tensor $T_{t(O)}^{\ r}$ is conserved in region O;

$$\partial_r T_{t(O)}^{\ r} = 0. \tag{2·16}$$

By integrating Eq. (2·16), we obtain

$$T_{t(O)}^{\ r} = a_o, \tag{2·17}$$

where a_o is an integration constant.

Finally, we consider the Ward identity for the general coordinate transformation in region H near the horizon. The Ward identity for the general coordinate transformation when there is a gravitational anomaly is

$$\nabla_\nu T_{\mu(H)}^{\ \nu}(r) - F_{\mu\nu} J_{(H)}^{\nu}(r) - \frac{\partial_\mu \Phi}{\sqrt{-g}} \frac{\delta S}{\delta \Phi} - \mathcal{A}_\mu(r) = 0, \tag{2·18}$$

where both the gauge current and the energy-momentum tensor are defined to be of the *covariant* form and $\mathcal{A}_\mu$ is the covariant form of the 2-dimensional gravitational anomaly. This Ward identity corresponds to that of Ref. 8) when there is no dilaton field. The covariant form of the 2-dimensional gravitational anomaly $\mathcal{A}_\mu$ is given by[14)–16)]

$$\mathcal{A}_\mu = \frac{1}{96\pi\sqrt{-g}} \epsilon_{\mu\nu} \partial^\nu R = \partial_r N_\mu^{\ r}, \tag{2·19}$$

where we define $N_\mu^{\ r}$ by

$$N_t^{\ r} \equiv \frac{ff" - (f')^2/2}{96\pi}, \qquad N_r^{\ r} \equiv 0, \tag{2·20}$$

and $\{\ '\ \}$ represents differentiation with respect to r. The $\mu = t$ component of (2·18) is written as

$$\partial_r T_{t(H)}^{\ r}(r) = F_{rt} J_{(H)}^{r}(r) + \partial_r N_t^{\ r}(r). \tag{2·21}$$

Using (2·10) and integrating (2·21) over r from r_+ to r, we obtain

$$T_{t(H)}^{\ r}(r) = -\frac{m^2}{2\pi} A_t(r_+) A_t(r) + \frac{m^2}{4\pi} A_t^2(r) + N_t^{\ r}(r) + \frac{m^2}{4\pi} A_t^2(r_+) - N_t^{\ r}(r_+), \tag{2·22}$$

where we impose the condition that the energy-momentum tensor vanishes at the horizon, which is the same as (2·11):

$$T_{t(H)}^{\ r}(r_+) = 0. \tag{2·23}$$

We compare (2·17) with (2·22). By following Banerjee and Kulkarni's approach,[10)] we impose the condition that the asymptotic form of (2·22) in the limit $r \to \infty$ is equal to (2·17):

$$T_{t(O)}^{\ r} = T_{t(H)}^{\ r}(\infty). \tag{2·24}$$

Condition (2·24) corresponds to the statement that no energy flux is generated away from the horizon region. Therefore, the asymptotic form of (2·22) has to agree with that of (2·17). From (2·24), we can obtain

$$a_o = \frac{m^2 \Omega^2}{4\pi} + \frac{\pi}{12\beta^2}, \tag{2·25}$$

where the quantity Ω is well known as the angular velocity in BH physics,[17)]

$$\Omega \equiv \frac{a}{r_+^2 + a^2}, \tag{2·26}$$

and we use both the surface gravity of the BH,

$$\kappa = \frac{2\pi}{\beta} = \frac{1}{2} f'(r_+), \tag{2·27}$$

and (2·20). As a result, we obtain the flux of the energy-momentum tensor in the region far from the horizon as

$$T^{\ r}_{t(O)} = \frac{m^2 \Omega^2}{4\pi} + \frac{\pi}{12\beta^2}. \tag{2·28}$$

This flux agrees with the Hawking flux. Our result corresponds to that of Ref. 7) in the limit $r \to \infty$. In contrast with Ref. 7), our result does not depend on gauge fields in the region far from the horizon when the radial coordinate r is large but finite. As can be seen from the action (2·2), the gauge field does not exist in the Kerr BH physics in a realistic 4-dimensional sense, and only the mass and angular momentum appear. We thus consider that our result presented here is more natural than that of Ref. 7).

§3. Conclusion and discussion

We have shown that the Ward identities and boundary conditions for covariant currents, without referring to the Wess-Zumino terms and the effective action, are sufficient to derive Hawking radiation. The first boundary condition states that both the $U(1)$ gauge current and the energy-momentum tensor vanish at the horizon, as in (2·11) and (2·23). This condition corresponds to the regularity condition that a free falling observer sees a finite amount of the charged current at the horizon. The second boundary condition is that the asymptotic form of the energy-momentum tensor defined in the region near the horizon is equal to the energy-momentum tensor in the region far from the horizon in the limit $r \to \infty$, as in (2·24). This condition means that no energy flux is generated far from the horizon. In contrast with previous works, we do not use the consistent current at any stage of our analysis since we use neither the Wess-Zumino term nor the effective action. We also do not use any step function. Therefore, we believe that our approach clarifies some essential aspects of the derivation of Hawking flux from anomalies.

When one compares our method with that of Iso et al.,[7)] one recognizes the following difference. They defined the gauge current by the φ component of the 4-dimensional energy-momentum tensor $T^{\ r}_{\varphi(4)}$ in the region far from the horizon. In contrast, we do not define the gauge current in the region far from the horizon, since no gauge current exists in a Kerr BH. This difference appears in (2·16), whereas Iso et al. used the equation

$$\partial_r T^{\ r}_{t(2)} - F_{rt} J^r_{(2)} = 0. \tag{3·1}$$

If we define gauge currents suitably, we might be able to consider the Kerr BH in the same way as the Reissner-Nordström BH, as Iso et al. attempt to do. However, some subtle aspects are involved in such methods that attempt to define gauge currents.

To be explicit, the authors in Ref. 7) regard part of the metrics as the gauge field by defining $A^\mu \equiv -g_{(4)}^{\mu\varphi}$, as in Kaluza-Klein theory. This definition is consistent with the initial definition of the current (2·6) near the horizon, i.e.,

$$A_t = \frac{g_{t\varphi(4)}}{g_{\varphi\varphi(4)}} = \frac{a(r^2+a^2-\triangle)}{a^2\triangle\sin^2\theta-(r^2+a^2)^2} \overset{\triangle\to 0}{\longrightarrow} -\frac{a}{r^2+a^2}. \tag{3·2}$$

To maintain consistency, they simultaneously assume that the definition of (2·15) is modified such that it leads to (3·1) by using the $\mu = t$ component of (2·12), i.e.,

$$T_{t(2)}^{\ r} = \int d\Omega_2 \left(r^2+a^2\cos^2\theta\right)\left(T_{t(4)}^{\ r} - A_t T_{\varphi(4)}^{\ r}\right). \tag{3·3}$$

In this way they maintain consistency. However, we consider that definition (2·15) is more natural than this modified definition, since in definition (2·15) the formal 2-dimensional energy-momentum tensor is defined by integrating the exact 4-dimensional energy-momentum tensor over the angular coordinates without introducing an artificial gauge current in the region far from the horizon. In our approach, which is natural for the Kerr BH, no gauge field appears in the region far from the horizon where the radial coordinate r is large but finite, in contrast to the formulation in Ref. 7). We thus believe that our formulation is more natural than the formulation in Ref. 7), although both formulations give rise to the same physical conclusion.

In passing, we mention that the Hawking flux is determined from (2·22) simply by considering the direct limit

$$T_{t(H)}^{\ r}(r\to\infty) = \frac{m^2}{4\pi}A_t^2(r_+) - N_t^{\ r}(r_+), \tag{3·4}$$

which agrees with (2·28). The physical meaning of this consideration is that Hawking radiation is induced by quantum anomalies, which are defined in an arbitrarily small region near the horizon since they are short-distance phenomena, and at any region far from the horizon the theory is anomaly-free and thus, no further flux is generated. Namely, we utilize an intuitive picture on the basis of the Gauss theorem, which is applied to a closed region surrounded by a surface S very close to the horizon and a surface S' far from the horizon in the asymptotic region (Fig. 2). If no flux is generated in this closed region, the flux on the surface very close to the horizon and the flux on the surface far from the horizon in the asymptotic region coincide.

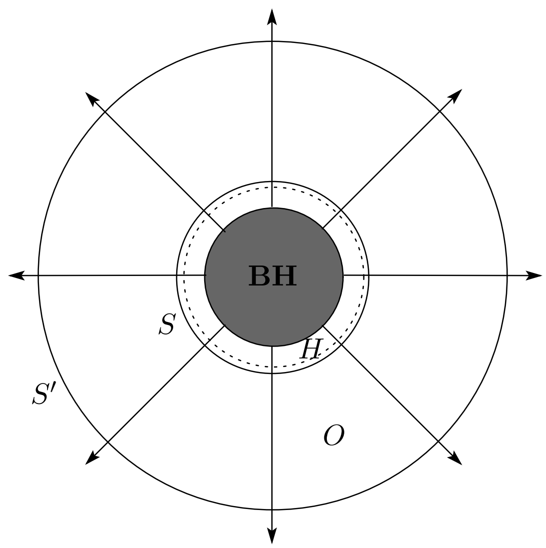

Fig. 2 Intuitive picture on the basis of the Gauss theorem.
The flux is only generated inside the dashed line. The total fluxes on S and S' are equal from the Gauss theorem.

Finally, we discuss why we use the regularity conditions for *covariant* currents instead of consistent currents. All the physical quantities should be gauge-invariant. Thus, physical currents should be *covariant*. This is consistent with, for example, the well-known anomalous baryon number current in the Weinberg-Salam theory.[19)]

For recent related works, please refer to Refs. 20)–22).

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Appendix A
—— *The Case for a Charged BH* ——

In this appendix, we show that Hawking flux can be obtained in a charged BH using our approach. Since we consider a charged Reissner-Nordström BH, the external space is given by the Reissner-Nordström metric

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2, \tag{A·1}$$

and $f(r)$ is given by

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = \frac{(r-r_+)(r-r_-)}{r^2}, \tag{A·2}$$

where $r_\pm = M \pm \sqrt{M^2 - Q^2}$ and r_+ is the distance from the center of the BH to the outer horizon. We consider quantum fields in the vicinity of the Reissner-Nordström BH. In 4 dimensions, the action for a complex scalar field is given by

$$S = \int d^4x \sqrt{-g} g^{\mu\nu} (\partial_\mu + ieA_\mu)\phi^* (\partial_\nu - ieA_\nu)\phi + S_{\text{int}}, \tag{A·3}$$

where the first term is the kinetic term and the second term S_{int} represents the mass, potential and interaction terms. In contrast with the Kerr BH background, note that the $U(1)$ gauge field $A_t = -Q/r$ appears in the Reissner-Nordström BH background. By performing the partial wave decomposition of ϕ in terms of the spherical harmonics ($\phi = \sum_{l,m} \phi_{lm} Y_{lm}$) and using the property $f(r_+) = 0$ at the horizon, the action S near the horizon is written as

$$S = -\sum_{l,m} \int dt dr r^2 \phi^*_{lm} \left[\frac{1}{f(r)} (\partial_t - ieA_t)^2 - \partial_r f(r) \partial_r \right] \phi_{lm}, \tag{A·4}$$

where we ignore S_{int} because the kinetic term dominates near the horizon in high-energy theory. From this action we find that ϕ_{lm} can be considered as $(1+1)$-dimensional complex scalar fields in the backgrounds of the dilaton $\varPhi$, metric $g_{\mu\nu}$ and $U(1)$ gauge field A_μ, where

$$\varPhi = r^2, \tag{A·5}$$

$$g_{tt} = f(r), \quad g_{rr} = -\frac{1}{f(r)}, \quad g_{rt} = 0, \tag{A·6}$$

$$A_t = -\frac{Q}{r}, \; A_r = 0. \tag{A·7}$$

The $U(1)$ charge of the 2-dimensional field ϕ_{lm} is e. Note that the action in the region far from the horizon is $S_{(O)}[\phi, g^{\mu\nu}_{(4)}, A_\mu]$ and the action in the region near the horizon is $S_{(H)}[\phi, g^{\mu\nu}_{(2)}, A_\mu, \varPhi]$.

We now present the derivation of Hawking radiation for the Reissner-Nordström BH. First, we consider the Ward identity for the gauge transformation in region O far away from the horizon. Here, we formally perform the path integral for $S_{(O)}[\phi, g^{\mu\nu}_{(4)}, A_\mu]$, where the Nöther current is constructed by the variational principle. Therefore, we can naturally treat *covariant* currents.[13] As a result, we obtain the Ward identity

$$\nabla_\mu J^\mu_{(4)} = 0, \tag{A·8}$$

where $J^\mu_{(4)}$ is the 4-dimensional gauge current. Since the Reissner-Nordström background is stationary and spherically symmetric, the expectation value of the gauge current in the background depends only on r, i.e., $\langle J^\mu \rangle = \langle J^\mu(r) \rangle$. Using the 4-dimensional metric, the conservation law (A·8) is written as

$$\partial_r (\sqrt{-g} J^r_{(4)}) + (\partial_\theta \sqrt{-g}) J^\theta_{(4)} = 0, \tag{A·9}$$

where $\sqrt{-g} = r^2 \sin\theta$. By integrating Eq. (A·9) over the angular coordinates θ and φ, we obtain

$$\partial_r J^r_{(2)} = 0, \tag{A·10}$$

where we define the effective 2-dimensional current $J^r_{(2)}$ by

$$J^r_{(2)} \equiv \int d\Omega_{(2)} r^2 J^r_{(4)}. \tag{A·11}$$

We define $J^r_{(2)} \equiv J^r_{(O)}$ to emphasize region O far from the horizon. The gauge current $J^r_{(O)}$ is conserved in region O,

$$\partial_r J^r_{(O)} = 0. \tag{A·12}$$

By integrating Eq. (A·12), we obtain

$$J^r_{(O)} = c_o, \tag{A·13}$$

where c_o is an integration constant.

Second, we consider the Ward identity for gauge transformation in the region H near the horizon. The Ward identity for the gauge transformation when there is a gauge anomaly is given by

$$\nabla_\mu J^\mu_{(H)} - \mathcal{B} = 0, \tag{A·14}$$

where we define the covariant current as $J^\mu_{(H)}$ and $\mathcal{B}$ is a covariant gauge anomaly. The covariant form of the 2-dimensional gauge anomaly $\mathcal{B}$ is given by

$$\mathcal{B} = \pm \frac{e^2}{4\pi\sqrt{-g}} \epsilon^{\mu\nu} F_{\mu\nu}, \qquad (\mu, \nu = t, r) \tag{A·15}$$

where $+(-)$ corresponds to the anomaly for right(left)-handed fields. Here $\epsilon^{\mu\nu}$ is an antisymmetric tensor with $\epsilon^{tr} = 1$ and $F_{\mu\nu}$ is the field-strength tensor. Using the 2-dimensional metric (A·6), (A·14) is written as

$$\partial_r J^r_{(H)}(r) = \frac{e^2}{2\pi} \partial_r A_t(r). \tag{A·16}$$

By integrating (A·16) over r from r_+ to r, we obtain

$$J^r_{(H)}(r) = \frac{e^2}{2\pi}[A_t(r) - A_t(r_+)], \tag{A·17}$$

where we impose the condition

$$J^r_{(H)}(r_+) = 0. \tag{A·18}$$

This condition corresponds to (2·11) in the present paper. We also impose the condition that the asymptotic form of (A·17) is equal to that of (A·13),

$$J^r_{(O)}(\infty) = J^r_{(H)}(\infty). \tag{A·19}$$

From (A·19), we obtain the gauge current in region O as

$$J^r_{(O)} = -\frac{e^2}{2\pi} A_t(r_+). \tag{A·20}$$

Third, we consider the Ward identity for the general coordinate transformation in the region O far from the horizon. We define the formal 2-dimensional energy-momentum tensor in region O from the exact 4-dimensional energy-momentum tensor in region O to connect the thus-defined 2-dimensional energy-momentum tensor in region O with the 2-dimensional energy-momentum tensor in region H. Since the action is $S_{(O)}[\phi, g^{\mu\nu}_{(4)}, A_\mu]$ in region O, the Ward identity for the general coordinate transformation is written as

$$\nabla_\nu T_{\mu(4)}^{\ \nu} - F_{\nu\mu} J^\nu_{(4)} = 0, \tag{A·21}$$

where $T^{\mu\nu}_{(4)}$ is the 4-dimensional energy-momentum tensor. Since the Reissner-Nordström background is stationary and spherically symmetric, the expectation value of the energy-momentum tensor in the background depends only on r, i.e., $\langle T^{\mu\nu}\rangle = \langle T^{\mu\nu}(r)\rangle$. The $\mu = t$ component of the conservation law (A·21) is written as

$$\partial_r \left(\sqrt{-g} T_{t(4)}^{\ r}\right) + \left(\partial_\theta \sqrt{-g}\right) T_{t(4)}^{\ \theta} - \sqrt{-g} F_{rt} J^r_{(4)} = 0. \tag{A·22}$$

By integrating (A·22) over θ and φ, we obtain

$$\partial_r T_{t(2)}^{\ r} = F_{rt} J^r_{(2)}, \tag{A·23}$$

where we define the effective 2-dimensional tensor $T_{t(2)}^{\ r}$ by

$$T_{t(2)}^{\ r} \equiv \int d\Omega_{(2)} r^2 T_{t(4)}^{\ r}, \tag{A·24}$$

and $J^r_{(2)}$ is defined by (A·11). To emphasize region O far from the horizon, we write (A·23) as

$$\partial_r T_{t(O)}^{\ r} = F_{rt} J^r_{(O)}. \tag{A·25}$$

By substituting (A·20) into (A·25) and integrating it over r, we obtain

$$T_{t(O)}^{\ r}(r) = a_o - \frac{e^2}{2\pi} A_t(r_+) A_t(r). \tag{A·26}$$

Finally, we consider the Ward identity for the general coordinate transformation in region H near the horizon. The Ward identity for the general coordinate transformation when there is a gravitational anomaly is

$$\nabla_\nu T_{\mu(H)}^{\ \nu} - F_{\nu\mu} J^\nu_{(H)} - \frac{\partial_\mu \Phi}{\sqrt{-g}} \frac{\delta S}{\delta \Phi} - \mathcal{A}_\mu = 0, \tag{A·27}$$

where both the gauge current and the energy-momentum tensor are defined to be of the *covariant* form and $\mathcal{A}_\mu$ is the covariant form of the 2-dimensional gravitational

anomaly. This Ward identity corresponds to that of Ref. 8) when there is no dilaton field. The covariant form of the 2-dimensional gravitational anomaly $\mathcal{A}_\mu$ agrees with (2·19). Using the 2-dimensional metric (A·6), the $\mu = t$ component of (A·27) is written as

$$\partial_r T^{\ r}_{t(H)}(r) = \partial_r \left[-\frac{e^2}{2\pi} A_t(r_+) A_t(r) + \frac{e^2}{4\pi} A_t^2(r) + N_t^{\ r} \right]. \qquad \text{(A·28)}$$

By integrating (A·28) over r from r_+ to r, we obtain

$$T^{\ r}_{t(H)}(r) = -\frac{e^2}{2\pi} A_t(r_+) A_t(r) + \frac{e^2}{4\pi} A_t^2(r) + N_t^{\ r}(r) + \frac{e^2}{4\pi} A_t^2(r_+) - N_t^{\ r}(r_+), \quad \text{(A·29)}$$

where we impose the condition that the energy-momentum tensor vanishes at the horizon, which is the same as (2·23):

$$T^{\ r}_{t(H)}(r_+) = 0. \qquad \text{(A·30)}$$

As for (2·24), we impose the condition that the asymptotic form of (A·29) in the limit $r \to \infty$ is equal to that of (A·26),

$$T^{\ r}_{t(O)}(\infty) = T^{\ r}_{t(H)}(\infty). \qquad \text{(A·31)}$$

From (A·31), we obtain

$$a_o = \frac{e^2}{4\pi} A_t^2(r_+) - N_t^{\ r}(r_+). \qquad \text{(A·32)}$$

We thus obtain the flux of the energy-momentum tensor in the region far from the horizon as

$$T^{\ r}_{t(O)}(r) = \frac{e^2 Q^2}{4\pi r_+^2} + \frac{\pi}{12\beta^2} + \frac{e^2 Q}{2\pi r_+} A_t(r). \qquad \text{(A·33)}$$

This result agrees with that of Ref. 5)*). In contrast with the case of a rotating Kerr BH, the energy flux depends on the gauge field in the region far from the horizon, but the radial coordinate r is still finite since the gauge field exists in a charged Reissner-Nordström BH background. However, in the evaluation of Hawking radiation by setting $r \to \infty$, the effect of the gauge field vanishes.

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*) For the generalization of the present analysis to higher-spin currents, see Ref. 20).

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